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AN EXACT CONSUMPTION-LOAN MODEL OF INTEREST WITH OR WITHOUT THE SOCIAL CONTRIVANCE OF MONEY*

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MY FIRST published paper[1] has come of age, and at a time when the subjects it dealt with have come back into fashion. It developed the equilibrium conditions for a rational consumer's lifetime consumption-saving pattern, a problem more recently given by Harrod the useful name of "hump saving" but which Landry, Böhm-Bawerk, Fisher, and others had touched on long before my time.[2] It dealt only with a single individual and did not discuss the mutual determination by all individuals of the market interest rates which each man had to accept parametrically as given to him.

Now I should like to give a complete general equilibrium solution to the determination of the time-shape of interest rates. This sounds easy, but actually it is very hard, so hard that I shall have to make drastic simplifications in order to arrive at exact results. For while Böhm and Fisher have given us the essential insights into the pure theory of interest, neither they nor other writers seem to have grappled with the following tough problem: in order to define an equilibrium path of interest in a perfect capital market endowed with *perfect certainty*, you have to determine *all* interest rates between now and the end of time; every finite time period points beyond itself!

Some interesting mathematical boundary problems, a little like those in the modern theories of dynamic programming, result from this analysis. And the way is paved for a rigorous attack on a simple model involving money as a store of value and a medium of exchange. My essay concludes with some provocative

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[1] "A Note on Measurement of Utility," *Review of Economic Studies*, IV (1937), 155–61.

[2] As an undergraduate student of Paul Douglas at Chicago, I was struck by the fact that we might, from the marginal utility schedule of consumptions, deduce saving behavior exactly in the same way that we might deduce gambling behavior. Realizing that, watching the consumer's gambling responses to varying odds, we could deduce his numerical marginal utilities, it occurred to me that, by watching the consumer's saving responses to varying interest rates, we might similarly measure his marginal utilities, and thus the paper was born. (I knew and pointed out, p. 155, n. 2; p. 160, that such a cardinal measurement of utility hinged on a certain refutable "independence" hypothesis.)

remarks about the field of social collusions, a subject of vital importance for political economy and of great analytical interest to the modern theorist.

THE PROBLEM STATED

Let us assume that men enter the labor market at about the age of twenty. They work for forty-five years or so and then live for fifteen years in retirement. (As children they are part of their parents' consumptions, and we take no note of them.) Naturally, they want to consume in their old age, and, in the absence of comprehensive social security—an institution which has important bearing on interest rates and saving—men will want to consume less than they produce during their working years so that they can consume something in the years when they produce nothing.

If there were only Robinson Crusoe, he would hope to put by some durable goods which could be drawn on in his old age. He would, so to speak, want to trade with Mother Nature current consumption goods in return for future consumption goods. And if goods kept perfectly, he could at worst always make the trade through time on a one-to-one basis, and we could say that the interest rate was zero ($i = 0$). If goods kept imperfectly, like ice or radium, Crusoe might have to face a negative real interest rate, $i < 0$. If goods were like rabbits or yeast, reproducing without supervision at compound interest, he would face a positive rate of interest, $i > 0$. This last case is usually considered to be technologically the most realistic one: that is, machines and round-about processes (rather than rabbits) are considered to have a "net productivity," and this is taken to be brute fact. (Böhm himself, after bitterly criticizing naïve productivity theorists and criticizing Thünen and others for assuming such a fact, ends up with his own celebrated third cause for interest, which also asserts the fact of net productivity. Contrary to much methodological discussion, there is nothing circular about assuming brute facts—that is all we can do; we certainly cannot deduce them, although, admittedly, we can hope by experience to refute falsely alleged facts.)

For the present purpose, I shall make the extreme assumption that nothing will keep at all. Thus no intertemporal trade with Nature is possible (that is, for all such exchanges we would have $i = -1$!). If Crusoe were alone, he would obviously die at the beginning of his retirement years.

But we live in a world where new generations are always coming along. Formerly we used to support our parents in their old age. That is now out of fashion. But cannot men during their productive years give up some of their product to bribe other men to support them in their retirement years? Thus, forty-year-old A gives some of his product to twenty-year-old B, so that when A gets to be seventy-five he can receive some of the product that B is then producing.

Our problem, then, is this: In a stationary population (or, alternatively, one growing in any prescribed fashion) what will be the intertemporal terms of trade or interest rates that will spring up spontaneously in ideally competitive markets?

SIMPLIFYING ASSUMPTIONS

To make progress, let us make convenient assumptions. Break each life up into thirds: men produce one unit of product in period 1 and one unit in period 2; in period 3 they retire and produce nothing. (No one dies in midstream.)

In specifying consumption preferences,

I suppose that each man's tastes can be summarized by an ordinal utility function of the consumptions of the three periods of his life: $U = U(C_1, C_2, C_3)$. This is the same in every generation and has the usual regular indifference-curve concavities, but for much of the argument nothing is said about whether, subjectively, men systematically discount future consumptions or satisfactions. (Thus Böhm's second cause of interest may or may not be operative; it could even be reversed, men being supposed to overvalue the future!)

In addition to ignoring Böhm's second cause of systematic time preference, I am in a sense also denying or reversing his first cause of interest, in that we are *not* supposing that society is getting more prosperous as time passes or that any single man can expect to be more prosperous at a later date in his life, since, on the contrary, during his years of retirement he must look forward to producing even less than during his working years.

Finally, recall our assumption that no goods keep, no trade with Nature being possible, and hence Böhm's third technological cause of interest is being denied.

Under these assumptions, what will be the equilibrium time path of interest rates?

INDIVIDUAL SAVING FUNCTIONS

The simplest case to tackle to answer this question is that of a stationary population, which has always been stationary in numbers and will always be stationary. This ideal case sidesteps the difficult "planning-until-infinity" aspect of the problem. In it births are given by $B_t = B$, the same constant for all positive and negative t.

Now consider any time t. There are B men of age one, B men of age two, and B retired men of age three. Since each producer produces 1 unit, total product is $B + B$. Now, for convenience of symbols, let $R_t = 1/(1 + i_t)$ be the discount rate between goods (chocolates) of period t traded for chocolates of the next period, $t + 1$. Thus, if $R_t = 0.5$, you must promise me two chocolates tomorrow to get me to part with one chocolate today, the interest rate being 100 per cent per period. If $R_t = 1$, the interest rate is zero, and tomorrow's chocolates cost 1.0 of today's. If $R_t > 1$, say $R_t = 1.5$, the interest rate is negative, and one future chocolate costs 1.5 of today's. (Clearly, R_t is the price of tomorrow's chocolates expressed in terms of today's chocolates as numeraire.)

We seek the equilibrium levels of . . . R_t, R_{t+1}, . . . , that will clear the competitive markets in which present and future goods exchange against each other.

At time t each man who is beginning his life faces[3] the budget equation,

$$C_1 + C_2R_t + C_3R_tR_{t+1} = 1 + 1R_t + 0R_tR_{t+1} . \quad (1)$$

This merely says that the total discounted value of his life's consumptions must equal the discounted value of his productions. Subject to this constraint, he will, for each given R_t and R_{t+1}, determine an optimal (C_1, C_2, C_3) to maximize $U(C_1, C_2, C_3)$, which we can summarize by the "demand" functions,

$$C_i = C_i(R_t, R_{t+1}) \qquad (i = 1, 2, 3) . \quad (2)$$

[3] I rule out, as I did explicitly in my 1937 paper (p.160), the Ulysses-Strotz-Allais phenomenon whereby time perspective distorts present decisions planned for the future from later actual decisions. Thus, if at the end of period 1 his ordinal preference follows $V(C_1, C_2, C_3)$ rather than $U(C_1, C_2, C_3)$, I am assuming $(\partial V/\partial C_i)/(\partial V/\partial C_j) \equiv (\partial U/\partial C_i)/(\partial U/\partial C_j)$. Hence all later decisions will ratify earlier plans. For a valuable discussion of this problem see R. H. Strotz, "Myopia and Inconsistency in Dynamic Utility Maximization," *Review of Economic Studies*, XXIII (1956), 165–80.

It might be convenient for us to work with "net" or "excess demands" of each man: these are the algebraic differences between what a man consumes and what he produces. Net demands in this sense are the negative of what men usually call "saving," and, in deference to capital theory, I shall work with such "net saving" as defined by

$$\begin{aligned} S_1 &= S_1(R_t, R_{t+1}) = 1 - C_1(R_t, R_{t+1}) , \\ S_2 &= S_2(R_t, R_{t+1}) = 1 - C_2(R_t, R_{t+1}) , \\ S_3 &= S_3(R_t, R_{t+1}) = 0 - C_3(R_t, R_{t+1}) . \end{aligned} \tag{3}$$

In old age presumably S_3 is negative, matched by positive youthful saving, so as to satisfy for all (R_t, R_{t+1}) the budget identity,

$$\begin{aligned} S_1(R_t, R_{t+1}) + R_t S_2(R_t, R_{t+1}) \\ + R_t R_{t+1} S_3(R_t, R_{t+1}) = 0 . \end{aligned} \tag{4}$$

Of course, these functions are subject to all the restrictions of modern consumption theory of the ordinal utility or revealed preference type. Thus, with consumption in every period being a "superior good," we can infer that $\partial C_3/\partial R_{t+1} > 0$ and $\partial S_3/\partial R_{t+1} < 0$. (This says that lowering the interest rate earned on savings carried over into retirement must increase retirement consumption.) We cannot unambiguously deduce the sign of $\partial S_1/\partial R_t$ and other terms, for the reasons implicit in modern consumption theory.

We can similarly work out the saving functions for men born a period later, which will be of the form $S_i(R_{t+1}, R_{t+2})$, etc., containing, of course, the later interest rates they will face—likewise for earlier interest rates facing men born earlier. Finally, our fundamental condition of clearing the market is this: Total net saving for the community must cancel out to zero in every period. (Remember that no goods keep and that real net investment is impossible, all loans being "consumption" loans.)

At any time t there exist B_t men of the first period, B_{t-1} men of the second period, and B_{t-2} men of the third period. The sum of their savings gives us the fundamental equilibrium condition:

$$\begin{aligned} 0 = B_t S_1(R_t, R_{t+1}) + B_{t-1} S_2(R_{t-1}, R_t) \\ + B_{t-2} S_3(R_{t-2}, R_{t-1}) , \end{aligned} \tag{5}$$

for every t. Note that in S_2 we have the interest rates of one earlier period than in S_1, and in S_3 we have still earlier interest rates (in fact, interest rates that are, at time t, already history and no longer to be determined.)

We have such an equation for every t, and if we take any finite stretch of time and write out the equilibrium conditions, we always find them containing discount rates from before the finite period and discount rates from afterward. We never seem to get enough equations: lengthening our time period turns out always to add as many new unknowns as it supplies equations, as will be spelled out later in equations (14).

THE STATIONARY CASE

We can try to cut the Gordian knot by our special assumption of stationariness, namely,

$$\begin{aligned} \ldots B_{t-1} = B_t = B_{t+1} = \ldots \\ = B, \text{ a given constant for all time} \\ \ldots R_{t-1} = R_t = R_{t+1} = \ldots \\ = R, \text{ the unknown discount rate.} \end{aligned} \tag{6}$$

The first of these is a demographic datum; the second assumption of nonchanging interest rates is a conjecture whose consistency we must explore and verify.

Now substituting relations (6) in equation (5), we get one equilibrium equation to determine our one unknown R, namely,

$$0 = BS_1(R, R) + BS_2(R, R) + BS_3(R, R) . \quad (7)$$

By inspection, we recognize a solution of equation (7) to be $R = 1$, or $i = 0$: that is, zero interest must be one equilibrium rate under our conditions.[4]

Why? Because

$$B[S_1(1, 1) + 1S_2(1, 1) + 1S_3(1, 1)] = 0$$

by virtue of the budget identity (4).

Can a common-sense explanation of this somewhat striking result be given? Let me try. In a stationary system everyone goes through the same life-cycle, albeit at different times. Giving over goods now to an older man is figuratively giving over goods to *yourself* when old. At what rate does one give over goods to one's later self? At $R > 1$, or $R < 1$, or $R = 1$? To answer this, note that a chocolate today *is* a chocolate today, and when middle-aged A today gives over a chocolate to old B, there is a one-to-one *physical* transfer of chocolates, none melting in the transfer and none sticking to the hands of a broker. So, heuristically, we see that the hypothetical "transfer *through time*" of the chocolates must be at $R = 1$ with the interest rate i exactly zero.

Note that this result is quite independent of whether or not people have a systematic subjective preference for present consumption over future. Why? Because we have assumed that if *anyone* has such a systematic preference, *everyone* has such a systematic preference. There is no one any different in the system, no outsider—so to speak—to exact a positive interest rate from the impatient consumers.[5]

[4] We shall see that $R = 1$ is not the only root of equation (7) and that there are multiple equilibriums.

A BIOLOGICAL THEORY OF INTEREST AND POPULATION GROWTH

A zero rate of population growth was seen to be consistent with a zero rate of interest for a consumption-loan world. I now turn to the case of a population growing exponentially or geometrically. Now

$$B_t = B(1 + m)^t , \qquad \text{with}$$

$$B_{t+1} = (1 + m)B_t = (1 + m)^2 B_{t-1} \ldots .$$

For $m > 0$, we have growth; for $m < 0$, decay; for $m = 0$, our previous case of a stationary population. As before, we suppose

$$\ldots R_{t-1} = R_t = R_{t+1} = \ldots = R, \text{ a constant through time.}$$

Now our clearing-of-the-market equation is

$$0 = B(1 + m)^t S_1(R, R) + B(1 + m)^{t-1} S_2(R, R) + B(1 + m)^{t-2} S_3(R, R) ; \quad (8)$$

or, cancelling $B(1 + m)^t$, we have

$$0 = S_1(R, R) + (1 + m)^{-1} S_2(R, R) + (1 + m)^{-2} S_3(R, R) . \quad (9)$$

Recalling our budget identity (4), we realize $R = (1 + m)^{-1}$ or $i = m$ is one root satisfying the equation, giving

$$0 = S_1(R, R) + RS_2(R, R) + R^2 S_3(R, R) .$$

We have therefore established the following paradoxical result:

[5] If productive opportunities were to exist, Mother Nature would operate as an important outsider, with whom trade could take place, and our conclusion would be modified. But recall our strong postulate that such technological opportunities are non-existent.

THEOREM: Every geometrically growing consumption-loan economy has an equilibrium market rate of interest exactly equal to its biological percentage growth rate.

Thus, if the net reproductive rate gives a population growth of 15 per cent per period, $i = 0.15$ is the corresponding market rate of interest. If, as in Sweden or Ireland, $m < 0$ and population decays, the market rate of interest will be negative, with $i < 0$ and $R > 1$!

OPTIMUM PROPERTY OF THE BIOLOGICAL INTEREST RATE

The equality of the market rate of interest in a pure consumption-loan world to the rate of population growth was deduced solely from mechanically finding a root of the supply-demand equations that clear the market. Experience often confirms what faith avers: that competitive market relations achieve some kind of an optimum.

Does the saving-consumption pattern given by $S_1(R, R)$, $S_2(R, R)$, $S_3(R, R)$, where $R = 1/(1 + m)$, represent some kind of a social optimum? One would guess that, if it does maximize something, this equilibrium pattern probably maximizes the "lifetime (ordinal) well-being of a representative person, subject to the resources available to him (and to every other representative man) over his lifetime." Or, what seems virtually the same thing, consider a cross-sectional family or clan that has an unchanging age distribution because the group remains in statistical equilibrium, though individuals are born and die. Such a clan will divide its available resources to maximize a welfare function differing only in scale from each man's utility function and will achieve the same result as the biological growth rate.

To test this optimality conjecture, first stick to the stationary population case. The representative man is thought to maximize $U(C_1, C_2, C_3)$, subject to

$$C_1 + C_2 + C_3 = 1 + 1\,, \qquad (10)$$

$1 + 1$ being the lifetime product available to each man. The solution to this technocratic welfare problem (free in its formulation and solution of all mention of prices or interest rates) requires

$$\frac{\partial U/\partial C_2}{\partial U/\partial C_1} = \frac{\partial U/\partial C_3}{\partial U/\partial C_1}. \qquad (11)$$

But this formulation is seen to be identical with that of a single maximizing man facing market discount rates $R_1 = R_2 = 1$. Hence the solution of equations (10) and (11) is exactly that given earlier by equation (3): that is, our present welfare problem has, for its optimality solution,

$$1 - C_1 = S_1(1, 1)\,,$$

$$1 - C_2 = S_2(1, 1)\,,$$

$$0 - C_3 = S_3(1, 1)\,.$$

Now that we have verified our conjecture for the stationary $m = 0$ case, we can prove it for population growing like $B(1 + m)^t$, where $m \gtrless 0$. As before, we maximize $U(C_1, C_2, C_3)$ for the representative man. But what resources are now available to him? Recall that in a growing population the age distribution is permanently skewed in favor of the younger productive ages: society and each clan has an age distribution proportional to $[1, 1/(1 + m), 1/(1 + m)^2]$ and has therefore a per capita output to divide in consumption among the three age classes satisfying

$$C_1 + \frac{1}{1+m} C_2 + \frac{1}{(1+m)^2} C_3 = 1 + \frac{1}{1+m}. \qquad (12)$$

By following a representative man throughout his life and remembering that there are always $(1+m)^{-1}$ just older than he and $(1+m)^{-2}$ two periods older, we derive this same "budget" or availibility equation. Subject to equation (12), we maximize $U(C_1, C_2, C_3)$ and necessarily end up with the same conditions as would a competitor facing the biological market interest rate $R_1 = R_2 = 1/(1+m)$: namely,

$$\begin{aligned} 1 - C_1 &= S_1(R, R)\,, \\ 1 - C_2 &= S_2(R, R)\,, \\ 0 - C_3 &= S_3(R, R)\,, \qquad (13) \\ R &= \frac{1}{1+m}\,. \end{aligned}$$

Hence the identity of the social optimality conditions and the biological market interest theory has been demonstrated.[6]

COMMON-SENSE EXPLANATION OF BIOLOGICAL MARKET INTEREST RATE

Productivity theorists have always related interest to the biological habits of rabbits and cows. And Gustav Cassel long ago developed a striking (but rather nonsensical) biological theory relating interest to the life-expectancy of men of means and their alleged propensity to go from maintaining capital to the buying of annuities at an allegedly critical positive i. I seem to be the first, outside a slave economy, to develop a biological theory of interest relating it to the reproductivity of human mothers.

Is there a common-sense market explanation of this (to me at least) astonishing result? I suppose it would go like this: in a growing population men of twenty outnumber men of forty; and retired men are outnumbered by workers more than in the ratio of the work span to the retirement span. With more workers to support them, the aged live better than in the stationary state—the excess being positive interest on their savings.

Such an explanation cannot be deemed entirely convincing. Outside of social security and family altruism, the aged have no claims on the young: cold and selfish competitive markets will not teleologically respect the old; the aged will get only what supply and demand impute to them.

So we might try another more detailed explanation. Recall that men of forty or of period 2 bargain with men of twenty or period 1, trying to bribe the latter to provide them with consumption in their retirement. (Men of over sixty-five or of period 3 can make fresh bargains with no one: after retirement it is too late for them to try to provide for their old age.) In a growing population there are more period 1 men for period 2 men to bargain with; this presumably confers a competitive advantage on period 2 men, the manifestation of it being the positive interest rate.

So might go the explanation. It is at least superficially plausible, and it does qualitatively suggest a positive interest rate when population is growing, al-

[6] If U has the usual quasi-concavity, this social optimum will be unique—whether U does or does not have the time-symmetry that is sometimes (for concreteness) assumed in later arguments. Not only will the representative man's utility U be maximized, but so will the "total" of social utility enjoyed over a long period of time: specifically, the divergence from attainable bliss

$$[U(C_1, C_2, C_3) - U^*] + [U(C_1, C_2, C_3) - U^*] + \ldots$$

over all time will be miminized, where U^* is the utility achieved when $R_1 = 1 = R_2$ and $S_i = S_i(1, 1)$. This theorem may require that we use an ordinal utility indicator that is concave in the C_i, as it is always open to us to do.

Of course, this entire footnote and the related text need obvious modifications if $m \neq 0$.

though perhaps it falls short of explaining the remarkable quantitative identity between the growth rates of interest and of population.

THE INFINITY PARADOX REVEALED

But will the explanation survive rigorous scrutiny? Is it true, in a growing or in a stationary population, that twenty-year-olds are, in fact, overconsuming so that the middle-aged can provide for their retirement? Specifically, in the stationary case where $R = 1$, is it necessarily true that $S_1(1, 1) < 0$? Study of $U(C_1, C_2, C_3)$ shows how doubtful such a general result would be; thus, if there is no systematic subjective time preference so that U is a function *symmetric* in its arguments, it would be easy to show that $C_1 = C_2 = C_3 = \frac{2}{3}$, with $S_1(1, 1) = S_2(1, 1) = +\frac{1}{3}$ and $S_3(1, 1) = -\frac{2}{3}$. Contrary to our scenario, the middle-aged are *not* turning over to the young what the young will later make good to them in retirement support.

THE TWO-PERIOD CASE

The paradox is delineated more clearly if we suppose but two equal periods of life—work and retirement. Now it becomes *impossible* for *any* worker to find a worker younger than himself to be bribed to support him in old age. Whatever the trend of births, there is but one equilibrium saving pattern possible: during working years, consumption equals product and saving is zero; the same during the brutish years of retirement. What equilibrium interest rate, or R, will prevail? Since no transactions take place, $R = 0/0$, so to speak, and appears rather indeterminate—and rather academic. However, if men desperately want *some* consumption at *all* times, only $R = \infty$ can be regarded as the (virtual) equilibrium rate, with interest equal to -100 per cent per period.[7]

We think we know the right answer just given in the two-period case. Let us test our previous mathematical methods. Now our equations are much as before and can be summarized by:

$$\text{Maximize } U(C_1, C_2) = U(1 - S_1, 0 - S_2)$$

$$\text{subject to } S_1 + R_t S_2 = 0 .$$

The resulting saving functions, $S_1(R_t)$ and $S_2(R_t)$, are subject to the budget identity,

$$S_1(R_t) + R_t S_2(R_t) \equiv 0 \text{ for all } R_t . \quad (4')$$

Clearing the market requires

$$0 = B_t S_1(R_t) + B_{t-1} S_2(R_{t-1}) \qquad \text{for} \quad t = 0, \pm 1, \pm 2, \ldots . \quad (5')$$

If $B_t = B(1 + m)^t$ and $R_t = R_{t+1} = \ldots = R$, our final equation becomes

$$0 = B\left[S_1(R) + \frac{1}{1+m} S_2(R)\right] . \quad (8')$$

The budget equation (4′) assures us that equation (8′) has a solution:

$$R = \frac{1}{1+m} \qquad \text{or} \qquad m = i .$$

$$\text{with} \qquad 0 < S_1(R) = -RS_2(R) .$$

So the two-period mathematics appears to give us the same answer as before—a biological rate of interest equal to the rate of population growth.

Yet we earlier deduced that *there can be no voluntary saving in a two-period world*. Instead of $S_1 > 0$, we must have $S_1 = 0 = S_2$ with $R = +\infty$. How can we reconcile this with the mathematics?

[7] A later numerical example, where $U = \log C_1 + \log C_2 + \log C_3$, shows that cases can arise where no positive R, however large, will clear the market. I adopt the harmless convention of setting $R = \infty$ in every case, even if the limit as $R \to \infty$ does not wipe out the discrepancy between supply and demand.

We substitute $S_1 = 0 = S_2$ in equation (5′) or equation (8′), and indeed this does satisfy the clearing-of-the-market equation. Apparently our one equilibrium equation in our one unknown R has more than a single solution! And the relevant one for a free market is *not* that given by our biological or demographic theory of interest, even though our earlier social optimality argument does perfectly fit the two-period case.

THE PARADOX CONTEMPLATED

The transparent two-period case alerts us to the possibility that in the three-period (or n-period) case, the fundamental equation of supply and demand may have multiple solutions. And, indeed, it does.[8] We see that

$$0 = S_1(\infty, \infty) = S_2(\infty, \infty) = S_3(\infty, \infty)$$

is indeed a valid mathematical solution. This raises the following questions:

Is a condition of no saving with dismal retirement consumption and interest rate of -100 per cent per period thinkable as the economically correct equilibrium for a free market?

Surely, the non-myopic middle-aged will do almost anything to make retirement consumption, C_3 non-zero?[9]

One might conjecture that the fact that, in the three-period model, workers can always find younger workers to bargain with is a crucial difference from the two-period case.[10] To investigate the problem, we must drop the assumption of a population that is, always has been, and always will be stationary (or exponentially growing or exponentially decaying). For within that ambiguous context $R = 1 (R < 1,\ R > 1)$ was indeed an impeccable solution, in the sense that no one can point to a violated equilibrium condition. (Exactly the same can be said of the two-period case, even though we "know" the impeccable solution is economically nonsense.)

We must give mankind a beginning. So, once upon a time, B men were born into the labor force. Then B more. Then B more. Until what? Until . . . ? Or until no more men are born? Must we give mankind an end as well as a beginning? Even the Lord rested after the beginning, so let us tackle one problem at a time and keep births forever constant. Our equilibrium equations, with the constant B's omitted, now become

$$\begin{aligned}
&S_1(R_1, R_2) + 0 + 0 = 0\,,\\
&S_2(R_1, R_2) + S_1(R_2, R_3) + 0 = 0\,,\\
&S_3(R_1, R_2) + S_2(R_2, R_3)\\
&\qquad\qquad + S_1(R_3, R_4) = 0\,,\\
&S_3(R_2, R_3) + S_2(R_3, R_4)\\
&\qquad\qquad + S_1(R_4, R_5) = 0\,,\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\,,\\
&S_3(R_{t-2}, R_{t-1}) + S_2(R_{t-1}, R_t)\\
&\qquad\qquad + S_1(R_t, R_{t+1}) = 0\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{14}$$

We feel that $S_1 \equiv 0 \equiv S_2 \equiv S_3$, while a mathematical solution, is not the economically relevant one. Since $S_1(1, 1)$, $S_2(1, 1)$, and $S_3(1, 1)$ do satisfy the last

[8] There is nothing surprising about multiple solutions in economics: not infrequently income effects make possible other intersections, including the possibility of an infinite number where demand and supply curves coincide.

[9] Before answering these questions, it would be well to decide what the word "surely" in the previous sentence means. Surely, no sentence beginning with the word "surely" can validly contain a question mark at its end? However, one paradox is enough for one article, and I shall stick to my economist's last.

[10] By introducing overlap between workers of different ages, the three-period model is essentially equivalent to a general n-period model or to the continuous-time model of real life.

of the written equations, we dare hope[11] that the Invisible Hand will ultimately work its way to the socially optimal biological-interest configuration—or that the solution to equation (14) satisfies

$$\lim_{t\to\infty} \quad R_t = 1, \quad S_i(R_t, R_{t+1}) = S_i(1, 1), \ (i = 1, 2, 3) . \tag{15}$$

THE IMPOSSIBILITY THEOREM

But have we any right to hope that the free market will even ultimately approach the specified social optimum? Does not the two-period case rob us of hope? Will not all the trade that the three-period case makes possible consist of middle-aged period 2 people giving consumption to young period 1 people in return for getting consumption back from them one period later? Do not such voluntary mutual-aid compacts suggest that, if R_t does approach a limit x, it must be such as to make $S_1(x, x) < 0$? Whereas, for many men[12] not too subject to systematic preference for the present over the future (not too affected by Böhm's second cause of interest), we expect $S_1(1, 1) > 0$.

A colleague, whose conjectures are often better than many people's theorems, has suggested to me that in the three-period or n-period case I am taking too bilateral a view of trade. We might end up with $S_1 > 0$ and encounter no contradictions to voluntary trade by virtue of the fact that young men trade with *anyone* in the market: they do not know or care that all or part of the motive for trade with them comes from the desire of the middle-aged to provide for retirement. The present young are content to be trading with the present old (or, for that matter, with the unborn or dead): all they care about is that their trades take place at the quoted market prices; and, if some kind of triangular or multilateral offsetting among the generations can take place and result in $S_1(R_t, R_{t+1})$ positive and becoming closer and closer to $S_1(1, 1) > 0$, why cannot this happen?

I, too, found the multilateral notion appealing. But the following considerations—of a type I do not recall seeing treated anywhere—suggest to me that the ultimate approach to $R = 1$ and $S_1(1, 1) > 0$ is quite impossible.

List all men from the beginning to time t. All the voluntary trades ever made must be mutually advantageous. If A gives something to B and B does nothing for A directly in return, we know B must be doing something for some C, who does do something good for A. (Of course, C might be more than one man, and there might be many-linked connections within C.)

Now consider a time when $S_1(R_t, R_{t+1})$ has become positive, with $S_2(R_{t-1}, R_t)$ also positive. Young man A is then giving goods to old man B. Young man A expects something in return and will actually two periods later be getting goods from someone. From whom? It certainly cannot be directly from B: B will

[11] Our confidence in this would be enhanced if the linear difference equation relating small deviations $r_t = R_t - 1$ had characteristic roots all less than 1 in absolute value. Thus $a_0 r_{t+3} + a_1 r_{t+2} + a_2 r_{t+1} + a_3 r_t = 0$, where the a_i are given in terms of the $S_i(R_t, R_{t+1})$ functions and their partial derivatives, evaluated at $R_t = 1$ R_{t+1}. Logically, this would be neither quite necessary nor sufficient: not sufficient, since the initial R_0, R_1, R_2 might be so far from 1 as to make the linear approximations irrelevant; not necessary, since, with one root less than unity in absolute value, we might ride in toward $R = 1$ on a razor's edge. In any case, as our later numerical example shows, our hope is a vain one.

[12] There is admittedly some econometric evidence that many young adults do dissave, to acquire assets and for other reasons. Some modifications of exposition would have to be made to allow for this.

be dead then. Let it be from someone called C. Can B ever do anything good for such a C, or have in the past done so? No. B only has produce during his first two periods of life, and all the good he can do anyone must be to people who were born before him or just after him. That *never* includes C. So the postulated pattern of $S_1 > 0$ is logically impossible in a free market: and hence $R_t = 1 = R_{t+1}$, as an exact or approximate relation, is impossible. (Note that, for some special pattern of time preference, the competitive solution *might* coincide with the "biological optimum.")

A NUMERICAL EXAMPLE

A concrete case will illustrate all this. The purest Marshallian case of unitary price and income elasticities can be characterized by $U = \log C_1 + \log C_2 + \log C_3$, where all systematic time preference is replaced by *symmetry*.

A maximum of

$$\sum_1^3 \log C_i \text{ subject to } C_1 + R_1 C_2 + R_1 R_2 C_3 = 1 + R_1 \tag{16}$$

implies

$$R_1 = \frac{\partial U/\partial C_2}{\partial U/\partial C_1} = \frac{1/C_2}{1/C_1},$$

$$R_1 R_2 = \frac{\partial U/\partial C_3}{\partial U/\partial C_1} = \frac{1/C_3}{1/C_1};$$

and, after combining this with the budget equation, we end up with saving functions,

$$\begin{aligned} S_1(R_1, R_2) &= \frac{2}{3} - \frac{R_1}{3}, \\ S_2(R_1, R_2) &= \frac{2}{3} - \frac{1}{3R_1}, \\ S_3(R_1, R_2) &= 0 - \frac{1}{3R_1R_2} - \frac{1}{3R_2}. \end{aligned} \tag{17}$$

Equations (14) now take the form

$$\begin{aligned} &\frac{2}{3} - \frac{R_1}{3} + 0 + 0 = 0, \\ &\frac{2}{3} - \frac{1}{3R_1} + \left(\frac{2}{3} - \frac{R_2}{3}\right) + 0 = 0, \\ &-\frac{1}{3R_1R_2} - \frac{1}{3R_2} + \left(\frac{2}{3} - \frac{1}{3R_2}\right) \\ &\qquad + \left(\frac{2}{3} - \frac{R_3}{3}\right) = 0, \\ &\left(-\frac{1}{3R_2R_3} - \frac{1}{3R_3}\right) + \left(\frac{2}{3} - \frac{1}{3R_3}\right) \\ &\qquad + \left(\frac{2}{3} - \frac{R_4}{3}\right) = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\left(-\frac{1}{3R_{t-1}R_{t-2}} - \frac{1}{3R_{t-1}}\right) \\ &\quad + \left(\frac{2}{3} - \frac{1}{3R_{t-1}}\right) + \left(\frac{2}{3} - \frac{R_t}{3}\right) = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{18}$$

Aside from initial conditions, this can be written in the recursive form,

$$R_t = 4 - \frac{1}{R_{t-1}R_{t-2}} - \frac{2}{R_{t-1}}. \tag{19}$$

Note that $\partial S_1(R_1, R_2) \equiv 0$ made our third-order difference equation degenerate into a second-order difference equation.

If we expand the last equation around $R_{t-2} = 1 = R_{t-1}$, retaining only linear terms and working in terms of deviations from the equilibrium level, $r_t = R_t - 1$, we get the recursive system,

$$r_{t+2} = 3r_{t+1} + r_t. \tag{20}$$

which obviously explodes away from $r = 0$ and $R = 1$ for all small perturbations from such an equilibrium. This confirms our proof that the *social optimum configuration can never here be reached by*

the competitive market, or even be approached in ever so long a time.

Where does the solution to (18) eventually go? Its first few R's are numerically calculated to be $[R_1, R_2, R_3, \ldots] = [2, 3\frac{1}{2}, 3\frac{2}{7}, \ldots]$. It is plain that the limiting R_t exceeds 1; hence a negative interest rate i is being asymptotically approached. Substituting $R_{t+2} = R_{t+1} = R_t = x$ in equation (19), we get the following cubic equation to solve for possible equilibrium levels:[13]

$$x = 4 - \frac{1}{x^2} - \frac{2}{x} \quad \text{or} \tag{21}$$

$$x^3 - 4x^2 + 2x + 1 = 0 .$$

We know that $x = 1$, the irrelevant optimal level, is one root; so, dividing it out, we end up with

$$(x - 1)(x^2 - 3x - 1) = 0 .$$

Solving the quadratic, we have

$$x = \frac{3 \pm \sqrt{9+4}}{2}$$

or

$$x = \frac{3}{2} + \frac{\sqrt{13}}{2} = 3.297 \text{ approx.}$$

for the asymptote approached by the free competitive market. The other root, $(3 - \sqrt{13})/2$, corresponds to a negative R, which is economically meaningless, in that it implies that the more we give up of today's consumption, the more we must give up of tomorrow's.

Our meaningful positive root, $R = 3.297$, corresponds to an ultimate negative interest rate,

$$i = \frac{1-R}{R} = -\frac{2.297}{3.297},$$

which implies that consumption loans lose about two-thirds of their principal in one period. This is here the competitive price to avoid retirement starvation.[14]

RECAPITULATION

The task of giving an exact description of a pure consumption-loan interest model is finished. We end up, in the stationary population case, with a negative market interest rate, rather than with the biological zero interest rate corresponding to the social optimum for the representative man. This was proved by the impossibility theorem and verified by an arithmetic example.

A corresponding result will hold for changing population where $m \gtrless 0$. The actual competitive market rate i_m will always be negative and always less than the biological optimality rate m.[15] And

[13] Martin J. Bailey has pointed out to me that the budget equation and the clearing-of-the-market equations do, in the stationary state, imply $S_1 = RS_3$ whenever $R \neq 1$, a fact which can be used to give an alternative demonstration of possible equilibrium values.

[14] In other examples, this competitive solution would not deviate so much from the $i = m$ biological optimum. But it is important to realize that solutions to equations (14) that come from quasi-concave utility functions—with or without systematic time preference—*cannot* be counted on to approach asymptotically the biological optimum configuration of equation (13).

In this case the linear approximation gives for $r_t = R_t - 3.297$ the recursion relation

$$r_{t+2} = \frac{1}{(3.297)^3} r_{t+1} + \frac{2}{(3.297)^2} r_t .$$

This difference equation has roots easily shown to be less than 1 in absolute value, so the local stability of our competitive equilibrium is assured.

[15] Writing $\lambda = 1/(1+m)$, our recursion relation (14) becomes

$$0 = S_1(R_t, R_{t+1}) + \lambda S_2(R_{t-1}, R_t) + \lambda^2 S_3(R_{t-2}, R_{t-1}) .$$

For the case where $U = \Sigma \log C_i$, our recursion relation (18) becomes

$$R_t = 2(1+\lambda) - \frac{\lambda^2}{R_{t-1}R_{t-2}} - \frac{\lambda^2}{R_{t-1}} - \frac{\lambda}{R_{t-1}} .$$

Then $x = R_t = R_{t-1} = R_{t-2}$ gives a cubic equation with biological root corresponding to $x = \lambda$ and

increasing the productive years relative to the retirement years of zero product would undoubtedly still leave us with a negative interest rate, albeit one that climbs ever closer to zero.

Is this negative interest rate a hard-to-believe result? Not, I think, when one recalls our extreme and purposely unrealistic assumptions. With Böhm's third technological reason for interest ruled out by assumption, with his second reason involving systematic preference for the present soft-pedaled, and with his first reason reversed (that is, with people expecting to be *poorer* in the future), we should perhaps have been surprised if the market rate had not turned out negative.

Yet, aside from giving the general biological optimum interest rate, our model is an instructive one for a number of reasons.

1. It shows us what interest rates would be implied if the "hump saving" process were acting alone in a world devoid of systematic time preference.[16]

2. It incidentally confirms what modern theorists showed long ago but what is still occasionally denied in the literature, that a zero or negative interest rate is in no sense a *logically* contradictory thing, however bizarre may be the *empirical* hypotheses that entail a zero or negative rate.

3. It may help us a little to isolate the effects of adding one by one, or together, (*a*) technological investment possibilities, (*b*) innovations that secularly raise productivity and real incomes, (*c*) strong biases toward present goods and against future goods, (*d*) governmental laws and more general collusions than are envisaged in simple laissez faire markets, or (*e*) various aspects of uncertainty. To be sure, other orderings of analysis would also be possible; and these separate processes interact, with the whole not the simple sum of its parts.

4. It points up a fundamental and intrinsic deficiency in a free pricing system, namely, that free pricing gets you on the Pareto-efficiency frontier but by itself has no tendency to get you to positions on the frontier that are ethically optimal in terms of a social welfare function; only by social collusions—of tax, expenditure, fiat, or other type—can an ethical observer hope to end up where he wants to be. (This obvious and ancient point is related to 3*d* above.)

5. The present model enables us to see one "function" of money from a new slant—as a social compact that can provide optimal old age social security. (This is also related to 3*d* above.)

For the rest of this essay, I shall develop aspects of the last two of these themes.

SOCIAL COMPACTS AND THE OPTIMUM

If each man insists on a *quid pro quo*, we apparently continue until the end of time, with each worse off than in the social optimum, biological interest case. Yet how easy it is by a simple change in the rules of the game to get to the optimum. Let mankind enter into a Hobbes-Rousseau social contract in which the

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$i = m$. The relevant competitive market root is given by

$$x = \frac{2+\lambda}{2} + \sqrt{\frac{(2+\lambda)^2+4\lambda}{2}}.$$

Where $m = 0$, $\lambda = 1$, we have $x = 3.297$; for $m \to \infty$, $\lambda \to 0$, $x \to 2$ and $i \to -\frac{1}{2}$; for $m \to -1$, $\lambda \to \infty$, $x \to \infty$ and $i \to -1$. Thus the market rate of interest is always between -1 and $-\frac{1}{2}$, growing as m grows, in agreement with the small husk of truth in our earlier "common-sense explanation."

[16] T. Ophir, of the Massachusetts Institute of Technology and Hebrew University, Jerusalem, has done unpublished work showing how systematic time preference will tend to alter the equilibrium interest rate pattern.

young are assured of their retirement subsistence if they will today support the aged, such support to be guaranteed by a draft on the yet-unborn. Then the social optimum can be achieved within one lifetime, and our equations (14) will become

$$S_1(1, 1) + S_2(1, 1) + S_3(1, 1) = 0$$

from $t = 3$ on.

We economists have been told[17] that what we are to economize on is love or altruism, this being a scarce good in our imperfect world. True enough, in the sense that we want what there is to go as far as possible. But it is also the task of political economy to point out where common rules in the form of self-imposed fiats can attain higher positions on the social welfare functions prescribed for us by ethical observers.

The Golden Rule or Kant's Categorical Imperative (enjoining like people to follow the common pattern that makes each best off) are often not self-enforcing: if all but one obey, the one may gain selfish advantage by disobeying—which is where the sheriff comes in: *we* politically invoke force on *ourselves*, attempting to make an unstable equilibrium a stable one.[18]

Once social coercion or contracting is admitted into the picture, the present problem disappears. The reluctance of the young to give to the old what the old can never themselves directly or indirectly repay is overcome. Yet the young never suffer, since their successors come under the same requirement. Everybody ends better off. It is as simple as that.[19]

The economics of social collusions is a rich field for analysis, involving fascinating predictive and normative properties. Thus, when society *acts as if* it were maximizing certain functions, we can predict the effect upon equilibrium of specified exogenous disburbances. And certain patterns of thought appropriate to a single mind become appropriate,

[17] D. H. Robertson, *What Does the Economist Maximize?* (a keynote address at the Columbia bicentennial celebrations, May, 1954), published by the Trustees of the University in the *Proceedings of the Conference, 1955* (New York: Doubleday & Co.) and reprinted as chap. ix in D. H. Robertson, *Economic Commentaries* (London: Staples, 1956).

[18] Now, admittedly, there is usually lacking in the real world the axes of symmetry needed to make all this an easy process. In a formulation elsewhere, I have shown some of the requirements for an optimal theory of public expenditure of the Sax-Wicksell-Lindahl-Musgrave-Bowen type, and the failure of the usual voting and signaling mechanisms to converge to an optimum solution (see "The Pure Theory of Public Expenditure," *Review of Economics and Statistics*, XXXVI [November, 1954], 387–89, and "Diagrammatic Exposition of Public Expenditure," *ibid.*, XXXVII [November, 1955], 350–56). Such a model is poles apart from the pure case in which Walrasian laissez faire happens to be optimal. I should be prepared to argue that a good deal of what is important and interesting in the real world lies between these extreme poles, perhaps in between in the sense of displaying properties that are a blending of the polar properties. But such discussion must await another time.

[19] How can the competitive configuration with negative interest rates be altered to everyone's advantage? Does not this deny the Pareto optimality of perfect competition, which is the least (and most) we can expect from it? Here we encounter one more paradox, which no doubt arises from the "infinity" aspect of our model. If we assume a large finite span to the human race—say 1 million generations—then the final few generations face the equations

$$S_1(R_{T-1}, \infty) + S_2(R_{T-2}, R_{T-1}) + S_3(R_{T-3}, R_{T-2}) = 0\,,$$

$$S_2(R_{T-1}, \infty) + S_3(R_{T-2}, R_{T-1}) + 0 = 0\,,$$

$$S_3(R_{T-1}, \infty) + 0 + 0 = 0\,,$$

$$\text{where} \quad T = 1{,}000{,}000\,.$$

If we depart from the negative interest rate pattern, the final young will be cheated by the demise of the human race. Should such a cheating of one generation 30 million years from now perpetually condemn society to a suboptimal configuration? Perfect competition shrugs its shoulders at such a question and (not improperly) sticks to its Pareto optimality.

even though we reject the notion of a group mind. (Example: developed social security could give rise to the same bias toward increasing population that exists among farmers and close family groups, where children are wanted as a means of old age support.)

The economics of collusion provides an important field of study for the theorist. Such collusions can be important elements of strength in the struggle for existence. Reverence for life, in the Schweitzer sense of respecting ants and flowers, might be a handicap in the Darwinian struggle for existence. (And, since the reverencer tends to disappear, the ants may not be helped much in the long run.) But culture in which altruism abounds—because men do not think to behave like atomistic competitors or because men have by custom and law entered into binding social contracts—may have great survival and expansion powers.

An essay could be written on the welfare state as a complicated device for self- or reinsurance. (From this view, the graduated income tax becomes in part a device for reducing *ex ante* variance.) That the Protestant Ethic should have been instrumental in creating individualistic capitalism one may accept; but that it should stop there is not necessarily plausible.[20] What made Jeremy Bentham a Benthamite in 1800, one suspects, might in 1900 have made him a Fabian (and do we not see a lot in common in the personalities of James Mill and Friedrich Engels?).

Much as you and I may dislike government "interferences" in economic life, we must face the positive fact that the motivations for higher living standards that a free market channels into Walrasian equilibrium when the special conditions for that pattern happen to be favorable—these same motivations often lead to social collusions and myriad uses of the apparatus of the state. For good or evil, these may not be aberrations from laissez faire, but theorems entailed by its intrinsic axioms.

[20] Recall the Myrdal thesis that the austere planned economies of Europe are Protestant, the Catholic countries being individualistic.

CONCLUSION: MONEY AS A SOCIAL CONTRIVANCE

Let me conclude by applying all these considerations to an analysis of the role of money in our consumption-loan world. In it nothing kept. All ice melted, and so did all chocolates. (If non-depletable land existed, it must have been superabundant.) Workers could not carry goods over into their retirement years.

There is no arguing with Nature. But what is to stop man—or rather men—from printing oblongs of paper or stamping circles of shell. These units of money can keep.[21] (Even if ink fades, this could be true.) With ideal clearing arrangements, money as a medium of exchange might have little function. But remember that a money medium of exchange is itself a rather efficient clearing arrangement.

So suppose men officially through the state, or unofficially through custom, make a grand consensus on the use of these greenbacks as a money of exchange. Now the young and middle-aged do have something to hold and to carry over into their retirement years. And note this: as long as the new current generations of

[21] I have been asked whether introducing durable money does not violate my fiat against durable goods and trades with Nature. All that I must insist on is that the new durable moneys (or records) be themselves quite worthless for consumption. The essence of them as money is that they are valued only for what they will fetch in exchange.

workers do not repudiate the old money, this gives workers of one epoch a claim on workers of a later epoch, even though no real *quid pro quo* (other than money) is possible.

We then find this remarkable fact: without legislating social security or entering into elaborate social compacts, society by using money will go from the non-optimal negative-interest-rate configuration to the optimal biological-interest-rate configuration. How does this happen? I shall try to give only a sketchy account that does not pretend to be rigorous.

Take the stationary population case with $m = 0$. With total money M constant and the flow of goods constant, the price level can be expected very soon to level off and be constant. The productive invest their hump savings in currency; in their old age they disinvest this currency, turning it over to the productive workers in return for sustenance.

With population growing like $(1 + m)^t$, output will come to grow at that rate. Fixed M will come to mean prices falling like $1/(1 + m)^t$. Each dollar saved today will thus yield a *real* rate of interest of exactly m per period—just what the biological social-optimality configuration calls for. Similarly, when $m < 0$ and population falls, rising prices will create the desired negative real rate of interest equal to m.

In short, the use of money can itself be regarded as a social compact.[22] When economists say that one of the functions of money is to act as a store of wealth and that one of money's desirable properties is constancy of value (as measured by constancy of average prices), we are entitled to ask: How do you know this? Why *should* prices be stable? On what tablets is that injunction written? Perhaps the function of money, if it is to serve as an optimal store of wealth, is so to change in its value as to create that optimal pattern of lifetime saving which could otherwise be established only by alternative social contrivances.[23]

I do not pretend to pass judgment on the policies related to all this. But I do suggest for economists' further research the difficult analysis of capital models which grapple with the fact that each and every today is followed by a tomorrow.

[22] In terms of immediate self-interest the existing productive workers should perhaps unilaterally repudiate the money upon which the aged hope to live in retirement. (Compare the Russian and Belgium calling-in of currencies.) So a continuing social compact is required. (Compare, too, current inflationary trends which do give the old less purchasing power than many of them had counted on.)

[23] Conversely, with satisfactory social security programs, the necessity for having secular stable prices so that the retired are taken care of can be lightened. Even after extreme inflations, social security programs can re-create themselves anew astride the community's indestructible real tax base.